## Definition 2.4

"If ***p***, then ***q***" is called the implication of ***p*** and ***q***, denoted as ***p***→***q***. ***p*** is called the antecedent (or hypothesis) of the implication, and ***q*** is called the consequent (or conclusion). The symbol → is called the ***implication connective***.

- It is defined that ***p***→***q*** is false if and only if p is true and q is false.

e.g. **Example:**

"If the weather is good tomorrow, we will go on an outing."

Let ***p***: The weather is good tomorrow,

***q***: We will go on an outing.

This can be formalized as ***p***→***q***.

- **Logical relationship of $p \to q$:**
  ***q*** is a necessary condition for ***p***, and ***p*** is a sufficient condition for ***q***.
- **Various ways to express "If *p*, then *q*" (all having the same truth value as $p \to q$):**
  (1) If ***p***, then ***q***.
  (2) If ***p***, just ***q***. (Emphasizes that the existence of ***p*** is a necessary condition for the existence of ***q***.)
  (3) ***p*** only if ***q***. (Means that for ***p*** to be true, ***q*** must also be true.)
  (4) Only if ***q***, then ***p***. (Indicates that the truth of ***q*** is a necessary condition for the truth of ***p***.)
  (5) Unless ***q***, not ***p***. (If ***q*** is not the case (***q*** is false), then ***p*** does not hold (***p*** is false).

## ■ Truth and Falsity of the Implication $p \to q$:

- When $p$ is true, $q$ must also be true for $p \to q$ to be "**true**".
- When $p$ is true and $q$ is false, $p \to q$ is "**false**".
- When $p$ is false, the implication does not impose any restrictions on the truth value of $q$, and $p \to q$ is "**true**".

*e.g.* »» **Example:**

Let $p$ be defined as "It is raining today," and $q$ be defined as "The ground is wet." The implication $p \to q$ can be described as "If it rains today, then the ground will be wet."

- In this case, the implication is **only false** if it rains (p is true) and the ground is not wet (q is false). In all other scenarios (such as it not raining, or it raining and the ground being wet), the implication is true.

## e.g.⋙ Example:

Let ***p***: It is cold, Let ***q***: Wang wears a down jacket.,

Here's the translation of the propositions with logical symbolism:

(1) As long as it is cold, Xiao Wang will wear a down jacket. $p \to q$

(2) Because it is cold, Xiao Wang wears a down jacket. $p \to q$

(3) If Xiao Wang does not wear a down jacket, then it is not cold. $\neg q \to \neg p$ or $p \to q$

(4) Only if it is cold, Xiao Wang will wear a down jacket. $q \to p$

## e.g.>>> Example:

Let $p$: It is cold, Let $q$: Wang wears a down jacket.

Here's the translation of the propositions with logical symbolism:

(5) Unless it is cold, Xiao Wang will wear a down jacket. $q \to p$

(6) Unless Xiao Wang wears a down jacket, otherwise it will not be cold. $p \to q$

(7) If it is not cold, then Xiao Wang will not wear a down jacket. $\neg p \to \neg q$ or $q \to p$

(8) Xiao Wang wears a down jacket only if it is cold. $q \to p$

## Definition 2.5:

The statement "***p*** if and only if ***q***" is called the equivalence of ***p*** and ***q***, denoted by ***p*↔*q***, where ↔ is called the ***biconditional operator***. It is defined that ***p*↔*q*** is true if and only if both ***p*** and ***q*** are true or both are false.

## Logical Relationship of p↔q:

***p*** and ***q*** are mutually sufficient and necessary conditions for each other.

e.g. **Example:**

Zhang San can do this task well, and only Zhang San can do it well.

Let ***p***: Zhang San does the task, ***q***: The task is done well.

This can be formalized as: ***p*↔*q***.

e.g. **Example:** Determine the truth value of the following compound propositions:

(1) 2 + 2 = 4 if and only if 3 + 3 = 6. 1

(2) 2 + 2 = 4 if and only if 3 is even. 0

(3) 2 + 2 = 4 if and only if the sun rises in the east. 1

(4) 2 + 2 = 5 if and only if the sun rises in the west. 1

(5) A necessary and sufficient condition for $f(x)$ to be differentiable at $x_0$ is that it is continuous at $x_0$. 0

## Truth Values of Basic Compound Propositions

| $p$ | $q$ | $p \wedge q$ | $p \vee q$ | $p \to q$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

## Precedence of Logical Connectives:

Parentheses (), Negation (¬), Conjunction (∧), Disjunction (∨), Implication (→), Biconditional (↔).

- **Same Level:** Evaluated from left to right.

- **Propositional Constants:** Simple propositions
- **Propositional Variables:** Variables that can take the value 0 (true) or 1 (false).
- **Definition 2.6 Well-Formed Formula (Propositional Formula, Formula):** A well-formed formula is **recursively defined** as follows:

(1) A single propositional constant or variable is a well-formed formula, also called an atomic formula.

(2) If A is a well-formed formula, then $(\neg A)$ is also a well-formed formula.

(3) If A and B are well-formed formulas, then $(A \land B)$, $(A \lor B)$, $(A \to B)$, and $(A \leftrightarrow B)$ are also well-formed formulas.

(4) Only those expressions formed by a finite number of applications of (1) through (3) are considered well-formed formulas.

e.g. **Example: 0, $p$, $\neg p \lor q$, $(p \lor q) \land (\neg p \lor r)$, $p \lor q \to r$, $(p \to q) \to r$**

## ■ Definition 2.7

(1) A single propositional variable or propositional constant is a 0-layer formula.

(2) A formula *A* is an (n+1)-layer formula (where n ≥ 0) if one of the following conditions is met:

① $A=\neg B$, where *B* is an n-layer formula.

② $A=B\wedge C$, where *B* and *C* are i-layer and j-layer formulas, respectively, and n = max(i, j).

③ $A=B\vee C$, where *B* and *C* are i-layer and j-layer formulas, respectively, and n = max(i, j).

④ $A=B\rightarrow C$, where *B* and *C* are i-layer and j-layer formulas, respectively, and n = max(i, j).

⑤ $A=B\leftrightarrow C$, where *B* and C are i-layer and j-layer formulas, respectively, and n = max(i, j).

e.g. **Example:** The propositional formulas

| | |
|---|---|
| $P$ | (0-layer) |
| $\neg p$ | (1-layer) |
| $\neg p \rightarrow q$ | (2-layer) |
| $(\neg(p \rightarrow q)) \leftrightarrow r$ | (3-layer) |
| $((\neg p \wedge q) \rightarrow r)) \leftrightarrow (\neg r \vee s)$ | (4-layer) |

## Definition 2.8

- Let $p_1, p_2, \ldots, p_n$ be all the propositional variables appearing in the formula $A$.
- Assigning a set of truth values to $p_1, p_2, \ldots, p_n$ is called an **assignment** or **interpretation** for $A$.
- An assignment that makes the formula true is called a ***satisfying assignment***, and an assignment that makes the formula false is called a ***falsifying assignment***.

## Definition 2.8 Explanation:

(1) An assignment is denoted as $\alpha=\alpha_1\alpha_2\ldots\alpha_n$, where each $\alpha_i$ is either 0 or 1, and the $\alpha_i$ are written without any punctuation marks between them.

(2) Generally, the assignment corresponds to the propositional variables in the order of their subscripts or alphabetical order. That is:

- When all propositional variables in $A$ are $p_1,p_2,\ldots,p_n$ , assigning $\alpha_1\alpha_2\ldots\alpha_n$ to A means $p_1=\alpha_1$, $p_2=\alpha_2,\ldots$, $p_n=\alpha_n$.
- When all propositional variables in $A$ are $p,q,r,\ldots$, assigning $\alpha_1\alpha_2\alpha_3\ldots$ to A means $p_1=\alpha_1$, $p_2=\alpha_2$, $p_3=\alpha_3$ ,… .

e.g. **Example:**

- Formula $A=(\neg p1 \wedge \neg p2 \wedge \neg p3) \vee (p1 \wedge p2)$
  - The assignment 000 is a **satisfying assignment** (makes the formula **true**).
  - The assignment 001 is a **falsifying assignment** (makes the formula **false**).
- Formula $B=(p \rightarrow q) \rightarrow r$
  - The assignment 000 is a **falsifying assignment** (makes the formula **false**).
  - The assignment 001 is a **satisfying assignment** (makes the formula **true**).

- **Truth Table:** A list of the values taken by a propositional formula under all possible assignments.
  - A formula with n variables has $2^n$ assignments.

e.g. **Example:** Provide the truth table for the following propositional formula.

(1) $(q \to p) \wedge q \to p$

| $p$ $q$ | $q \to p$ | $(q \to p) \wedge q$ | $(q \to p) \wedge q \to p$ |
|---|---|---|---|
| 0 0 | 1 | 0 | 1 |
| 0 1 | 0 | 0 | 1 |
| 1 0 | 1 | 0 | 1 |
| 1 1 | 1 | 1 | 1 |

e.g. **Example:** Provide the truth table for the following propositional formula.

(2) $\neg(\neg p \vee q) \wedge q$

| $p$ $q$ | $\neg p$ | $\neg p \vee q$ | $\neg(\neg p \vee q)$ | $\neg(\neg p \vee q) \wedge q$ |
|---|---|---|---|---|
| 0 0 | 1 | 1 | 0 | 0 |
| 0 1 | 1 | 1 | 0 | 0 |
| 1 0 | 0 | 0 | 1 | 0 |
| 1 1 | 0 | 1 | 0 | 0 |

e.g. **Example:** Provide the truth table for the following propositional formula.

(3) $(p \vee q) \to \neg r$

| $p$ $q$ $r$ | $p \vee q$ | $\neg r$ | $(p \vee q) \to \neg r$ |
|---|---|---|---|
| 0 0 0 | 0 | 1 | 1 |
| 0 0 1 | 0 | 0 | 1 |
| 0 1 0 | 1 | 1 | 1 |
| 0 1 1 | 1 | 0 | 0 |
| 1 0 0 | 1 | 1 | 1 |
| 1 0 1 | 1 | 0 | 0 |
| 1 1 0 | 1 | 1 | 1 |
| 1 1 1 | 1 | 0 | 0 |

- **Tautology (Always True):** A propositional formula that is never false under any assignment.
- **Contradiction (Always False):** A propositional formula that is never true under any assignment.
- **Satisfiable Formula:** A propositional formula that is not a contradiction.

**Note:** A tautology is satisfiable, but the converse is not true.

**Examples:**

e.g. (1) $(q \to p) \land q \to p$ is a tautology.

(2) $\neg(\neg p \lor q) \land q$ is a contradiction.

(3) $(p \lor q) \to \neg r$ is a satisfiable formula that is not a tautology.

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- **2.2.1 Equivalence Expressions and Equivalence Calculus**
  - Equivalence Expressions and Basic Equivalence Expressions
  - Truth Table Method and Equivalence Calculus Method
- **2.2.2 Connective Complete Set**
  - Truth Functions
  - Connective Complete Set
  - NAND Connective, NOR Connective

- **Definition 2.11:**

  Let $A$ and $B$ be two propositional formulas. If the equivalence expression $A \leftrightarrow B$ is a tautology (Universally Valid Formula), then $A$ and $B$ are said to be equivalent, denoted as $A \Leftrightarrow B$, and called an ***equivalence expression***.

- **Explanation:**

(1) $\Leftrightarrow$ is the notation for equivalence, which is different from the equivalence connective $\leftrightarrow$.

(2) $A \Leftrightarrow B$ means that propositions $A$ and $B$ are either both "true" or both "false" under all possible assignments (i.e., they have the same truth table).

(3) Every propositional formula has infinitely many equivalent propositional formulas(e.g.,: $\neg\neg P$ is equivalent to $P$).

## ↳ Equivalence Expressions (cont.)



### Explanation:

(3) Every propositional formula has infinitely many equivalent propositional formulas(e.g.,: ¬¬*P* is equivalent to *P*).

(4) In propositional logic, "Equivalence" is a more commonly used and clearer term used to indicate that two propositions have the same truth value under all possible conditions. This is similar to the concept of "Equality".

(5) There may be dummy variables in *A* or *B*.

For example, in $(p \to q) \leftrightarrow ((\neg p \vee q) \vee (\neg r \wedge r))$, *r* is a dummy variable in the left-hand formula.

The value of a dummy variable does not affect the truth value of the propositional formula.

## ↳ Determine Equivalence Expression Using a Truth Table



e.g. Example： Judge $\neg(p\vee q)$ and $\neg p\wedge\neg q$ is equal or not.

Solve：

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p\vee q$ | $\neg(p\vee q)$ | $\neg p\wedge\neg q$ | $\neg(p\vee q)\leftrightarrow(\neg p\wedge\neg q)$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

Conclusion: $\neg(p\vee q) \Leftrightarrow (\neg p\wedge\neg q)$